3.4.4

Using the Lagrange multipliers method and setting $g(x,y) = x^2 - y^2$, we have to find points (x,y,z) that satisfy both

$$g(x,y) = 2$$

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

This means

$$x^2 - y^2 = 2$$

$$1 = 2\lambda x$$

$$-1 = -2\lambda y$$

The last two equations give $x = y$, but then the first one never holds. Thus, there are no critical points constrained to the given hyperbola.

3.4.6

Using the same method and setting $g(x,y,z) = x^2 - y^2$ and $h(x,y,z) = 2x + z$ we have to solve in this case

$$g(x,y,z) = 1$$

$$h(x,y,z) = 1$$

$$\nabla f(x,y,z) = \alpha \nabla g(x,y,z) + \beta \nabla h(x,y,z)$$

Which means

$$x^2 - y^2 = 1$$

$$2x + z = 1$$

$$1 = 2x\alpha + 2\beta$$

$$1 = -2y\alpha$$

$$1 = \beta$$

After substituting β by 1 we get that the 3rd equation becomes $-1 = 2x\alpha$, which in combination with the 4th equation gives $x = y$, which never satisfies the 1st equation. Hence f has no local extrema in this restriction.

3.4.20

If the dimensions of the box are x, y and z meters, then the surface S and the volume V are given by

$$S(x, y, z) = xy + 2yz + 2xz$$

$$V(x, y, z) = xyz$$

Since $x \geq 0$, $y \geq 0$ and $z \geq 0$, the possible values of x, y, z form a compact surface and thus V has a global maximum and minimum. Since in the boundary all the values for V are 0, we just have to check the critical points.

The equations become

$$S(x, y, z) = 16$$

$$\nabla V(x, y, z) = \lambda \nabla S(x, y, z)$$

Which means

$$xy + 2yz + 2xz = 16$$

$$yz = \lambda(y + 2z)$$

$$xz = \lambda(x + 2z)$$

$$xy = \lambda(2x + 2y)$$

Dividing the second by the third equation we have

$$\frac{y}{x} = \frac{y + 2z}{x + 2z} \iff xy + 2yz = xy + 2xz \iff x = y$$

Where the last equation holds since $z \neq 0$ because we are looking for interior points.

Now substituting $y = x$ in the last equation we have $\lambda = x/4$, and then looking at the third equation we have

$$xz = \frac{x(x + 2z)}{4} \iff z = \frac{x + 2z}{4} \iff z = x/2$$

Substituting this in the original equation we have

$$3x^2 = 16 \Rightarrow x = \frac{4\sqrt{3}}{3}$$

Which gives the point $p = \left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{6}\right)$, and thus $V(p) = \frac{32\sqrt{3}}{9}$

3.4.31

a) By definition, the gradient $\nabla f(x)$ must satisfy

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - \nabla f(x) \cdot h}{\|h\|} = 0$$

Now using that $(Ax) \cdot x = x^T Ax$ we have:

$$\lim_{h \to 0} \frac{(x+h)^T A(x+h) - x^T Ax - 2\nabla f(x) \cdot h}{\|h\|} = 0$$

Putting $(x+h)^T A(x+h) = x^T Ax + h^T Ah + 2h^T Ax$ we have

$$\lim_{h \to 0} \frac{h^T Ah + 2h^T Ax - 2\nabla f(x) \cdot h}{\|h\|} = 0$$

We can see $h^T Ah/\|h\|$ as $(h/\|h\|)^T Ah$, and hence this term already goes to 0, since Ah approaches 0 as well. Therefore, the gradient must satisfy

$$\lim_{h \to 0} \frac{2h^T Ax - 2\nabla f(x) \cdot h}{\|h\|} = 0$$

Which gives $\nabla f(x) = (Ax)^T$. (Remember that the gradient is a row vector)

Note that this solves the general case, but since the matrix is 3×3 the gradient could be computed by hand.

b) Since a sphere has no 1-dimensional boundary, its maximum and minimum must be attained at critical points restricted to the sphere. Hence altogether with the restriction $g(x) = 1$, where $g(x) = x \cdot x = \|x\|^2$, it must hold the Lagrange multiplier equation:

$$\nabla f(x) = \lambda \nabla g(x)$$

Which means $(Ax)^T = \lambda(2x)^T$ as desired. Note that having a 2 at the right side doesn't affect the existence of λ.

c) Since we are now in a compact 3-dimensional surface, the candidates for maximum and minimum are the interior points in which $\nabla f(x) = 0$ or the points in the sphere in which $Ax = \lambda x$ (by part (b)).

$\nabla f(x) = 0$ means $Ax = 0$ and therefore $f(x) = 0$. However, $Ax = \lambda x$ means $f(x) = (\lambda/2)x \cdot x = \lambda/2$, since x is in the unit sphere. This gives that the candidates are 0 along with the eigenvalues of A divided by 2, and thus the minimum of f in the unit ball is the minimum of these numbers, and the maximum of f is the maximum of these numbers.

3.4.38

We have to find the maximum of $f(x, y) = xy - x - y + 1$ under the restriction $g(x, y) \leq B$, where $g(x, y) = xp + yq$.

If we check at the interior points of the region $g(x, y) \leq B$, $x \geq 1$ and $y \geq 1$, the only candidate is the point $(1, 1)$, when the gradient vanishes, but this gives $f(1, 1) = 0$, and the same happens for points in the boundary $x = 1$ or $y = 1$ [note that $f(x, y) = (x - 1)(y - 1)$].

Then, the candidates are given by the solutions of $\nabla f(x, y) = \lambda g(x, y)$ and the restriction g with equality, which means

$$y - 1 = \lambda p$$

$$x - 1 = \lambda q$$

$$xp + yq = B$$

Substituting in the third equation $x = 1 + \lambda q$ and $y = 1 + \lambda p$ we get

$$p + q + 2\lambda pq = B \Rightarrow \lambda = \frac{B - p - q}{2pq}$$

And plugging back in the first two equations we get

$$x = \frac{B + p - q}{2p}$$

$$y = \frac{B + q - p}{2q}$$

The quotient of these gives the ratio.

3.5.4

Treating x as a constant we get the solutions of the quadratic equation

$$y = \frac{2 \pm \sqrt{4 - 4x(x^2 + 2)}}{2x}$$

(a) All the points in which the discriminant $4 - 4x(x^2 + 2)$ is positive, there is a neighborhood in which it remains positive, and hence we can solve for y in a neighborhood. The only point in which this doesn't happen is when the discriminant is 0, where we can solve for y but not in a neighborhood, those points satisfy

$$y = \frac{2}{2x} \iff xy = 1$$

(b) Applying the implicit function theorem, we know that we can isolate y around a neighborhood at points satisfying $\frac{\partial F(x, y)}{\partial y} \neq 0$, which means

$$2yx - 2 \neq 0 \iff xy \neq 1$$

Finally, we can compute the value of dy/dx using the identity

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-y^2 - 2x}{2xy - 2}$$

3.5.7

If $F(x, y, z) = x^3 z^2 - z^3 yx$, then

$$\frac{\partial F}{\partial z}(1, 1, 1) = -1$$

But

$$\frac{\partial F}{\partial z}(0,0,0) = 0$$

This shows that we can solve for z near $(1,1,1)$ but not around $(0,0,0)$.

Again we have

$$\frac{dz}{dx} = \frac{-F_x}{F_z}(1,1,1) = 2$$

And

$$\frac{dz}{dx} = \frac{-F_x}{F_z}(1,1,1) = -1$$

3.5.10

Using the Inverse Function Theorem, we know that $F(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$ can be solved around a point (x,y,z) iff $|Df(x,y,z)| \neq 0$.

We have

$$|Df(0,0,0)| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 1$$

Which means that it can be solved around $(0,0,0)$.

3.5.14

Given $g(x,y,z) = x^2 + y^2 + z^2$, we have that

$$\frac{\partial g}{\partial x}(1,0,0) = 2$$

$$\frac{\partial g}{\partial y}(1,0,0) = 0$$

$$\frac{\partial g}{\partial z}(1,0,0) = 0$$

This means that at $(1,0,0)$ we can just solve the equation for x. The same applies for $(-1,0,0)$. An analogous reasoning is used for the other intersection points between the sphere and the axis.